

ON SEMI-MODULAR SUBALGEBRAS OF LIE ALGEBRAS OVER FIELDS OF ARBITRARY CHARACTERISTIC

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Abstract

This paper is a further contribution to the extensive study by a number of authors of the subalgebra lattice of a Lie algebra. It is shown that, in certain circumstances, including for all solvable algebras, for all Lie algebras over algebraically closed fields of characteristic $p > 0$ that have absolute toral rank ≤ 1 or are restricted, and for all Lie algebras having the one-and-a-half generation property, the conditions of modularity and semi-modularity are equivalent, but that the same is not true for all Lie algebras over a perfect field of characteristic three. Semi-modular subalgebras of dimensions one and two are characterised over (perfect, in the case of two-dimensional subalgebras) fields of characteristic different from 2, 3.

Keywords: Lie algebra; subalgebra lattice; modular; semi-modular, quasi-ideal.

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1 Introduction

This paper is a further contribution to the extensive study by a number of authors of the subalgebra lattice of a Lie algebra, and is, in part, inspired by the papers of Varea ([15], [16]). A subalgebra U of a Lie algebra L is called

- *modular* in L if it is a modular element in the lattice of subalgebras of L ; that is, if

$$\langle U, B \rangle \cap C = \langle B, U \cap C \rangle \quad \text{for all subalgebras } B \subseteq C,$$

and

$$\langle U, B \rangle \cap C = \langle B \cap C, U \rangle \quad \text{for all subalgebras } U \subseteq C,$$

(where, $\langle U, B \rangle$ denotes the subalgebra of L generated by U and B);

- *upper modular* in L (um in L) if, whenever B is a subalgebra of L which covers $U \cap B$ (that is, such that $U \cap B$ is a maximal subalgebra of B), then $\langle U, B \rangle$ covers U ;
- *lower modular* in L (lm in L) if, whenever B is a subalgebra of L such that $\langle U, B \rangle$ covers U , then B covers $U \cap B$;
- *semi-modular* in L (sm in L) if it is both um and lm in L .

In this paper we extend the study of sm subalgebras started in [12]. In section two we give an example of a Lie algebra over a perfect field of characteristic three which has a sm subalgebra that is not modular. However, it is shown that for all solvable Lie algebras, and for all Lie algebras over an algebraically closed field of characteristic $p > 0$ that have absolute toral rank ≤ 1 or are restricted, the conditions of modularity, semi-modularity and being a quasi-ideal are equivalent. The latter extends results of Varea in [16] where the characteristic of the field is restricted to $p > 7$. It is then shown that for all Lie algebras having the one-and-a-half generation property the conditions of modularity and semi-modularity are equivalent.

In section three, sm subalgebras of dimension one are studied. These are characterised over fields of characteristic different from 2, 3. This result generalises a result of Varea in [15] concerning modular atoms. In the fourth section we show that, over a perfect field of characteristic different from 2, 3, the only Lie algebra containing a two-dimensional core-free sm subalgebra is $sl_2(F)$. It is also shown that, over certain fields, every sm subalgebra that is solvable, or that is split and contains the normaliser of each of its non-zero subalgebras, is modular.

Throughout, L will denote a finite-dimensional Lie algebra over a field F . There will be no assumptions on F other than those specified in individual results. The symbol ' \oplus ' will denote a vector space direct sum. If U is a subalgebra of L , the *core* of U , U_L , is the largest ideal of L contained in U ; we say that U is *core-free* if $U_L = 0$. We denote by $R(L)$ the solvable radical of L , by $Z(L)$ the centre of L , and put $C_L(U) = \{x \in L : [x, U] = 0\}$.

2 General results

We shall need the following result from [12].

Lemma 2.1 *Let U be a proper sm subalgebra of a Lie algebra L over an arbitrary field F . Then U is maximal and modular in $\langle U, x \rangle$ for all $x \in L \setminus U$.*

Proof: We have that U is maximal in $\langle U, x \rangle$, by Lemma 1.4 of [12], and hence that U is modular in $\langle U, x \rangle$, by Theorem 2.3 of [12]

In [12] it was shown that, over fields of characteristic zero, U is modular in L if and only if it is sm in L . This result does not extend to all fields of characteristic three, as we show next. Recall that a simple Lie algebra is *split* if it has a splitting Cartan subalgebra H ; that is, if the characteristic roots of $\text{ad}_L h$ are in F for every $h \in H$. Otherwise we say that it is *non-split*.

Proposition 2.2 *Let L be a Lie algebra of dimension greater than three over an arbitrary field F , and suppose that every two linearly independent elements of L generate a three-dimensional non-split simple Lie algebra. Then there are maximal subalgebras M_1, M_2 of L such that $M_1 \cap M_2 = 0$.*

Proof: This is proved in Proposition 4 of [8].

Example

Let G be the algebra constructed by Gein in Example 2 of [7]. This is a seven-dimensional Lie algebra over a certain perfect field F of characteristic three. In G every linearly independent pair of elements generate a three-dimensional non-split simple Lie algebra. It follows from Proposition 2.2 above that there are two maximal subalgebras M, N in G such that $M \cap N = 0$. Choose any $0 \neq a \in M$. Then $\langle a, N \rangle \cap M = M$, but $\langle N \cap M, a \rangle = Fa$, so Fa is not a modular subalgebra of L . However, it is easy to see that all atoms of G are sm in G .

A subalgebra Q of L is called a *quasi-ideal* of L if $[Q, V] \subseteq Q + V$ for every subspace V of L . It is easy to see that quasi-ideals of L are always semi-modular subalgebras of L . When L is solvable the semi-modular subalgebras of L are precisely the quasi-ideals of L , as the next result, which is based on Theorem 1.1 of [15], shows.

Theorem 2.3 *Let L be a solvable Lie algebra over an arbitrary field F and let U be a proper subalgebra of L . Then the following are equivalent:*

- (i) U is modular in L ;
- (ii) U is sm in L ; and
- (iii) U is a quasi-ideal of L .

Proof: (i) \Rightarrow (ii) : This is straightforward.

(ii) \Rightarrow (iii) : Let L be a solvable Lie algebra of smallest dimension containing a subalgebra U which is sm in L but is not a quasi-ideal of L . Then U is maximal and modular in L , by Lemma 2.1, and $U_L = 0$. Let A be a minimal ideal of L . Then $L = U + A$. Moreover, $U \cap A$ is an ideal of L , since A is abelian, whence $U \cap A = 0$ and $L = U \oplus A$. Now U is covered by $\langle U, A \rangle$ so A covers $U \cap A = 0$. This yields that $\dim(A) = 1$ and so U is a quasi-ideal of L , a contradiction.

(iii) \Rightarrow (i) : This is straightforward.

Corollary 2.4 *Let L be a solvable Lie algebra over an arbitrary field F and let U be a core-free sm subalgebra of L . Then $\dim(U) = 1$ and L is almost abelian.*

Proof: This follows from Theorem 2.3 and Theorem 3.6 of [1].

We now consider the case when L is not necessarily solvable. First we shall need the following result concerning $psl_3(F)$.

Proposition 2.5 *Let F be a field of characteristic 3 and let $L = psl_3(F)$. Then L has no maximal sm subalgebra.*

Proof: Let E_{ij} be the 3×3 matrix that has 1 in the (i, j) -position and 0 elsewhere, and denote by $\overline{E_{ij}}$ the canonical image of $E_{ij} \in sl_3(F)$ in $psl_3(F)$. Put $e_{-3} = \overline{E_{23}}$, $e_{-2} = \overline{E_{31}}$, $e_{-1} = \overline{E_{12}}$, $e_0 = \overline{E_{11}} - \overline{E_{22}}$, $e_1 = \overline{E_{21}}$, $e_2 = \overline{E_{13}}$, $e_3 = \overline{E_{32}}$. Then $e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3$ is a basis for $psl_3(F)$ with

$$[e_0, e_i] = e_i \text{ if } i > 0, \quad [e_0, e_i] = -e_i \text{ if } i < 0, \quad [e_{-i}, e_j] = \delta_{ij}e_0 \text{ if } i, j > 0 \text{ and}$$

$$[e_i, e_j] = e_{-k} \text{ for every cyclic permutation } (i, j, k) \text{ of } (1, 2, 3) \text{ or } (-3, -2, -1).$$

Put $B_{i,j} = Fe_0 + Fe_i + Fe_j$ for each non-zero i, j . If i, j are of opposite sign then $B_{i,j}$ is a subalgebra, every maximal subalgebra of which is two dimensional.

Let M be a maximal sm subalgebra of L . For each i, j of opposite sign, if $B_{i,j} \not\subseteq M$ then $M \cap B_{i,j}$ is two dimensional. Since M is at most five-dimensional, by considering the intersection with each of $B_{1,-1}, B_{2,-2}$ and $B_{3,-3}$ it is easy to see that $e_0 \in M$. But then, considering $B_{1,-1}$ again, we have either $e_1 \in M$ or $e_{-1} \in M$. Suppose the former holds. Taking the intersection of M with $B_{2,-3}$ shows that $e_{-3} \in M$; then with $B_{2,-1}$ gives $e_2 \in M$; next with $B_{3,-2}$ gives $e_{-2} \in M$; finally with $B_{3,-1}$ yields $e_3 \in M$.

But then $M = L$, a contradiction. A similar contradiction is easily obtained if we assume that $e_{-1} \in M$.

Let $(L_p, [p], \iota)$ be any finite-dimensional p -envelope of L . If S is a subalgebra of L we denote by S_p the restricted subalgebra of L_p generated by $\iota(S)$. Then the (*absolute*) *toral rank* of S in L , $TR(S, L)$, is defined by

$$TR(S, L) = \max\{\dim(T) : T \text{ is a torus of } (S_p + Z(L_p))/Z(L_p)\}.$$

This definition is independent of the p -envelope chosen (see [11]). We write $TR(L, L) = TR(L)$. Then, following the same line of proof, we have an extension of Lemma 2.1 of [16].

Lemma 2.6 *Let L be a Lie algebra over an algebraically closed field of characteristic $p > 0$ such that $TR(L) \leq 1$. Then the following are equivalent:*

- (i) U is modular in L ;
- (ii) U is sm in L ; and
- (iii) U is a quasi-ideal of L .

Proof: We need only show that (ii) \Rightarrow (iii). Let U be a sm subalgebra of L that is not a quasi-ideal of L . Then there is an $x \in L$ such that $\langle U, x \rangle \neq U + Fx$. We have that U is maximal and modular in $\langle U, x \rangle$, by Lemma 2.1, and $\langle U, x \rangle$ is not solvable, by Theorem 2.3. Furthermore $TR(\langle U, x \rangle) \leq TR(L) \leq 1$, by Proposition 2.2 of [11], and $\langle U, x \rangle$ is not nilpotent so $TR(\langle U, x \rangle) \neq 0$, by Theorem 4.1 of [11], which yields $TR(\langle U, x \rangle) = 1$. We may therefore suppose that U is maximal and modular in L , of codimension greater than one in L , and that $TR(L) = 1$.

Put $L^\infty = \bigcap_{n \geq 1} L^n$. Suppose first that $R(L^\infty) \not\leq U$. Then $U \cap R(L^\infty)$ is maximal and modular in the solvable subalgebra $R(L^\infty)$, so $U \cap R(L^\infty)$ has codimension one in $R(L^\infty)$. Since U is maximal in L we have $L = U + R(L^\infty)$ and so $\dim(L/U) = 1$, which is a contradiction. This yields that $R(L^\infty) \leq U$. Moreover, $L^\infty \not\leq U$, since this would imply that U/L^∞ is maximal in the nilpotent algebra L/L^∞ , giving $\dim(L/U) = 1$, a contradiction again. It follows that $(U \cap L^\infty)/R(L^\infty)$ is modular and maximal in $L^\infty/R(L^\infty)$. But now $L^\infty/R(L^\infty)$ is simple, by Theorem 2.3 of [17], and $1 = TR(L) \geq TR(L^\infty, L) \geq TR(L^\infty/R(L^\infty))$ by section 2 of [11], so $TR(L^\infty/R(L^\infty)) = 1$. This implies that

$$p \neq 2, \quad L^\infty/R(L^\infty) \in \{sl_2(F), W(1 : \underline{1}), H(2 : \underline{1})^{(1)}\} \text{ if } p > 3$$

and $L^\infty/R(L^\infty) \in \{sl_2(F), psl_3(F)\}$ if $p = 3$,

by [9] and [10].

Now $H(2 : \underline{1})^{(1)}$ has no modular and maximal subalgebras, by Corollary 3.5 of [15]; likewise $psl_3(F)$ by Proposition 2.5. It follows that $L^\infty/R(L^\infty)$ is isomorphic to $W(1 : \underline{1})$, which has just one proper modular subalgebra and this has codimension one, by Proposition 2.3 of [15], or to $sl_2(F)$ in which the proper modular subalgebras clearly have codimension one. Hence $\dim(L^\infty/(U \cap L^\infty)) = 1$. Since $L = U + L^\infty$ we conclude that $\dim(L/U) = \dim(L^\infty/(U \cap L^\infty)) = 1$. This contradiction gives the claimed result.

We then have the following extension of Theorem 2.2 of [16]. The proof is virtually as given in [16], but as the restriction to characteristic > 7 has been removed the details need to be checked carefully. The proof is therefore included for the convenience of the reader.

Theorem 2.7 *Let L be a restricted Lie algebra over an algebraically closed field F of characteristic $p > 0$, and let U be a proper subalgebra of L . Then the following are equivalent:*

- (i) U is modular in L ;
- (ii) U is sm in L ; and
- (iii) U is a quasi-ideal of L .

Proof: As before it suffices show that (ii) \Rightarrow (iii). Let U be a sm subalgebra of L that is not a quasi-ideal of L . Then there is an $x \in L$ such that $\langle U, x \rangle \neq U + Fx$. First note that $\langle U, x \rangle$ is a restricted subalgebra of L . For, suppose not and pick $z \in \langle U, x \rangle_p$ such that $z \notin \langle U, x \rangle$. Since $\langle U, x \rangle$ is an ideal of $\langle U, x \rangle_p$ we have that $[z, U] \leq \langle U, x \rangle \cap \langle U, z \rangle$. But U is maximal in $\langle U, z \rangle$, by Lemma 2.1, and so $\langle U, x \rangle \cap \langle U, z \rangle = U$, giving $[z, U] \leq U$. But U is self-idealizing, by Lemma 1.5 of [12], so $z \in U$. This contradiction proves the claim. So we may as well assume that $L = \langle U, x \rangle$. Moreover, U is restricted since it is self-idealizing, whence $(U_L)_p \leq U$. As $(U_L)_p$ is an ideal of L we have that $U_L = (U_L)_p$. It follows that L/U_L is also restricted. We may therefore assume that U is a core-free modular and maximal subalgebra of L of codimension greater than one in L .

Now L is spanned by the centralizers of tori of maximal dimension, by Corollary 3.11 of [17], so there is such a torus T with $C_L(T) \not\leq U$. Let $L = C_L(T) \oplus \sum L_\alpha(T)$ be the decomposition of L into eigenspaces with

respect to T . We have that $C_L(T)$ is a Cartan subalgebra of L , by Theorem 2.14 of [17]. It follows from the nilpotency of $C_L(T)$ and the modularity of U that $U \cap C_L(T)$ has codimension one in $C_L(T)$.

Now let $L^{(\alpha)} = \sum_{i \in P} L_{i\alpha}(T)$, where P is the prime field of F , be the 1-section of L corresponding to a non-zero root α . From the modularity of U we see that $U \cap L^{(\alpha)}$ is a modular and maximal subalgebra of $L^{(\alpha)}$. Since U is core-free and self-idealizing, $Z(L) = 0$. But then $TR(T, L) = TR(L)$, since T is a maximal torus, whence $TR(L^{(\alpha)}) \leq 1$, by Theorem 2.6 of [11]. It follows from Lemma 2.6 that $M \cap L^{(\alpha)}$ is a quasi-ideal of $L^{(\alpha)}$. As $U \cap L^{(\alpha)}$ is maximal in $L^{(\alpha)}$, we have that $\dim(L^{(\alpha)}/(U \cap L^{(\alpha)})) \leq 1$ and $L^{(\alpha)} = U \cap L^{(\alpha)} + C_L(T)$. This yields that $L = U + C_L(T)$ and hence that $\dim(L/U) = \dim(C_L(T)/(U \cap C_L(T))) = 1$, a contradiction. The result follows.

We shall say that the Lie algebra L has the *one-and-a-half generation property* if, given any $0 \neq x \in L$, there is an element $y \in L$ such that $\langle x, y \rangle = L$. Then we have the following result.

Theorem 2.8 *Let L be a Lie algebra, over any field F , which has the one-and-a-half generation property. Then every sm subalgebra of L is a modular maximal subalgebra of L .*

Proof: Let U be a sm subalgebra of L and let $0 \neq u \in U$. Then there is an element $x \in L$ such that $L = \langle u, x \rangle = \langle U, x \rangle$. It follows from Lemma 2.1 that U is modular and maximal in L .

Corollary 2.9 *Let L be a Lie algebra over an infinite field F of characteristic different from 2, 3 which is a form of a classical simple Lie algebra. Then every sm subalgebra of L is a modular maximal subalgebra of L .*

Proof: Under the given hypotheses L has the one-and-a-half generation property, by Theorem 2.2.3 and section 1.2.2 of [3], or by [5].

We also have the following analogue of a result of Varea from [15].

Corollary 2.10 *Let F be an infinite perfect field of characteristic $p > 2$, and assume that $p^n \neq 3$. Then the subalgebra $W(1 : \mathbf{n})_0$ is the unique sm subalgebra of $W(1 : \mathbf{n})$.*

Proof: Let $L = W(1 : \mathbf{n})$ and let Ω be the algebraic closure of F . Then $L \otimes_F \Omega$ is simple and has the one-and-a-half generation property, by Theorem

4.4.8 of [3]. It follows that L has the one-and-a-half generation property (see section 1.2.2 of [3]). Let U be a sm subalgebra of L . Then U is modular and maximal in L by Theorem 2.8. Suppose that $U \neq L_0$. Then $L = U + L_0$ and $U \cap L_0$ is maximal in L_0 . But L_0 is supersolvable (see Lemma 2.1 of [13] for instance) so $\dim(L_0/(L_0 \cap U)) = 1$. It follows that $\dim(L/U) = \dim(L_0/(L_0 \cap U)) = 1$, whence $U = L_0$, which is a contradiction.

3 Semi-modular atoms

We say that L is *almost abelian* if $L = L^2 \oplus Fx$ with $\text{ad } x$ acting as the identity map on the abelian ideal L^2 . A μ -*algebra* is a non-solvable Lie algebra in which every proper subalgebra is one dimensional. A subalgebra U of a Lie algebra L is a *strong ideal* (respectively, *strong quasi-ideal*) of L if every one-dimensional subalgebra of U is an ideal (respectively, quasi-ideal) of L ; it is *modular** in L if it satisfies a dualised version of the modularity conditions, namely

$$\langle U, B \rangle \cap C = \langle B, U \cap C \rangle \quad \text{for all subalgebras } B \subseteq C,$$

and

$$\langle U \cap B, C \rangle = \langle B, C \rangle \cap U \quad \text{for all subalgebras } C \subseteq U.$$

Example

Let K be the three-dimensional Lie algebra with basis a, b, c and multiplication $[a, b] = c$, $[b, c] = b$, $[a, c] = a$ over a field of characteristic two. Then K has a unique one-dimensional quasi-ideal, namely Fc . Thus for each $0 \neq u \in Fc$ and $k \in K \setminus Fc$ we have that $\langle u, k \rangle$ is two dimensional. However K is not almost abelian. In fact K is simple, Fc is core-free and is the Frattini subalgebra of K , and so any two linearly independent elements not in Fc generate K .

We shall need a result from [4]. However, because of the above example, there is a (slight) error in three results in this paper. The error comes from an incorrect use of Theorem 3.6 of [1]. The three corrected results are as follows:

Lemma 3.1 (*Lemma 2.2 of [4]*) *If Q is a strong quasi-ideal of L , then Q is a strong ideal of L , or L is almost abelian, or F has characteristic two, $L = K$ and $Q = Fc$.*

Proof: Assume that Q is a strong quasi-ideal and that there exists $q \in Q$ such that Fq is not an ideal of L . Then Theorem 3.6 of [1] gives that L is almost abelian, or F has characteristic two, $L = K$ and $Q = Fc$. The result follows.

The proof of the following result is the same as the original.

Proposition 3.2 (*Proposition 2.3 of [4]*) *Let Q be a proper quasi-ideal of a Lie algebra L which is modular* in L . Then Q is a strong quasi-ideal and so is given by Lemma 3.1.*

Lemma 3.3 (*Lemma 4.1 of [4]*) *Let L be a Lie algebra over an arbitrary field F . Let U be a core-free subalgebra of L such that $\langle u, z \rangle$ is either two dimensional or a μ -algebra for every $0 \neq u \in U$ and $z \in L \setminus U$. Then one of the following holds:*

- (i) L is almost abelian;
- (ii) $\langle u, z \rangle$ is a μ -algebra for every $0 \neq u \in U$ and $z \in L \setminus U$;
- (iii) F has characteristic two, $L = K$ and $Fu = Fc$.

Proof: This is the same as the original proof except that the following should be inserted at the end of sentence six: “or $\text{char} F = 2$ and $L = K$ ”.

Using the above we now have the following result.

Lemma 3.4 *Suppose that Fu is sm in L but not an ideal of L . Then one of the following holds:*

- (i) L is almost abelian;
- (ii) $\langle u, x \rangle$ is a μ -algebra for every $x \in L \setminus Fu$;
- (iii) F has characteristic two, $L = K$ and $Fu = Fc$.

Proof: Pick any $x \in L \setminus Fu$. Then Fu is maximal in $\langle u, x \rangle$, by Lemma 2.1. Now let M be a maximal subalgebra of $\langle u, x \rangle$. If $u \in M$ then $M = Fu$. So suppose that $u \notin M$. Then Fu is a maximal subalgebra of $\langle u, x \rangle = \langle u, M \rangle$, whence $Fu \cap M = 0$ is maximal in M , since Fu is lm in L . It follows that every maximal subalgebra of $\langle u, x \rangle$ is one dimensional. The claimed result now follows from Lemma 3.3.

We shall need the following result concerning ‘one-and-a-half generation’ of rank one simple Lie algebras over infinite fields of characteristic $\neq 2, 3$.

Theorem 3.5 *Let L be a rank one simple Lie algebra over an infinite field F of characteristic $\neq 2, 3$ and let Fx be a Cartan subalgebra of L . Then there is an element $y \in L$ such that $\langle x, y \rangle = L$.*

Proof. Since L is rank one simple it is central simple. Let Ω be the algebraic closure of F and put $L_\Omega = L \otimes_F \Omega$, and so on. Then L_Ω is simple and Ωx is a Cartan subalgebra of L_Ω . Let

$$L_\Omega = \Omega x \oplus \sum_{\alpha \in \Phi} (L_\Omega)_\alpha$$

be the decomposition of L_Ω into its root spaces relative to Ωx . Then, with the given restrictions on the characteristic of the field, every root space $(L_\Omega)_\alpha$ is one dimensional (see [2]).

Let M be a maximal subalgebra of L containing x . Then M_Ω is a subalgebra of L_Ω and $\Omega x \subseteq M_\Omega$. So, M_Ω decomposes into root spaces relative to Ωx ,

$$M_\Omega = \Omega x \oplus \sum_{\alpha \in \Delta} (M_\Omega)_\alpha.$$

We have that $\Delta \subseteq \Phi$ and $(M_\Omega)_\alpha \subseteq (L_\Omega)_\alpha$ for all $\alpha \in \Delta$. As $(L_\Omega)_\alpha$ is one dimensional for every $\alpha \in \Phi$, we have $(M_\Omega)_\alpha = (L_\Omega)_\alpha$ for every $\alpha \in \Delta$. Hence there are only finitely many maximal subalgebras of L containing x : M_1, \dots, M_r say. Since F is infinite, $\cup_{i=1}^r M_i \neq L$, so there is an element $y \in L$ such that $y \notin M_i$ for all $1 \leq i \leq r$. But now $\langle x, y \rangle = L$, as claimed.

If U is a subalgebra of L , then the *normaliser* of U in L is the set

$$N_L(U) = \{x \in L : [x, U] \subseteq U\}.$$

We can now give the following characterisation of one-dimensional semi-modular subalgebras of Lie algebras over fields of characteristic $\neq 2, 3$.

Theorem 3.6 *Let L be a Lie algebra over a field F , of characteristic $\neq 2, 3$ if F is infinite. Then Fu is sm in L if and only if one of the following holds:*

- (i) Fu is an ideal of L ;
- (ii) L is almost abelian and $\text{ad } u$ acts as a non-zero scalar on L^2 ;
- (iii) L is a μ -algebra.

Proof: It is easy to check that if (i), (ii), or (iii) hold then Fu is sm in L . So suppose that Fu is sm in L , but that (i), (ii) do not hold. First we claim that L is simple.

Suppose not, and let A be a minimal ideal of L . If $u \in A$, choose any $b \in L \setminus A$. Then $\langle u, b \rangle \cap A$ is an ideal of $\langle u, b \rangle$. Since $0 \neq u \in \langle u, b \rangle \cap A$ and $b \notin A$, $\langle u, b \rangle$ cannot be a μ -algebra. But then L is almost abelian, by Lemma 3.4, a contradiction. So $u \notin A$. By Lemma 3.3 of [12], $ua = \lambda a$ for all $a \in A$ and some $\lambda \in F$. But now $Fu + Fa$ is a two-dimensional subalgebra of $\langle u, a \rangle$, a μ -algebra, which is impossible. Hence L is simple.

Now Fu is um in L and not an ideal of L , so $N_L(Fu) = Fu$, by Lemma 1.5 of [12]. Hence Fu is a Cartan subalgebra of L , and L is rank one simple. Now F cannot be finite, since there are no μ -algebras over finite fields, by Corollary 3.2 of [6]. Hence F is infinite. But then there is an element $y \in L$ such that $\langle u, y \rangle = L$, by Theorem 3.5, and L is a μ -algebra. The result is established.

As a corollary to this we have a result of Varea, namely Corollary 2.3 of [14].

Corollary 3.7 (*Varea*) *Let L be a Lie algebra over a perfect field F , of characteristic $\neq 2, 3$ if F is infinite. If Fu is modular in L but not an ideal of L then L is either almost abelian or three-dimensional non-split simple.*

Proof: This follows from Theorem 3.6 and the fact that with the stated restrictions on F the only μ -algebras are three-dimensional non-split simple (Proposition 1 of [7]).

4 Semi-modular subalgebras of higher dimension

First we consider two-dimensional semi-modular subalgebras. We have the following analogue of Theorem 1.6 of [15].

Theorem 4.1 *Let L be a Lie algebra over a perfect field F of characteristic different from 2, 3, and let U be a two-dimensional core-free sm subalgebra of L . Then $L \cong sl_2(F)$.*

Proof: If U is modular then the result follows from Theorem 1.6 of [15], so we can assume that U is not a quasi-ideal of L . Thus, there is an element $x \in L$ such that $\langle U, x \rangle \neq U + Fx$. Put $V = \langle U, x \rangle$. Then $U_V = U$ implies that $\langle U, x \rangle = U + Fx$, a contradiction; if $U_V = 0$ then $V \cong sl_2(F)$ by Lemma

2.1 and Theorem 1.6 of [15], and $\langle U, x \rangle = U + Fx$, a contradiction. It follows that $\dim(U_V) = 1$. Put $U_V = Fu$. Now $\dim(U/U_V) = 1$ and V/U_V is three-dimensional non-split simple, by Theorem 3.6 and Proposition 1 of [7]. Thus $V = Fu \oplus S$, where S is three-dimensional non-split simple, by Lemma 1.4 of [15], and Fu, S are ideals of V .

Now we claim that $0 \neq Z(\langle U, y \rangle) \subseteq U$ for every $y \in L \setminus U$. We have shown this above if $\langle U, y \rangle \neq U + Fy$. So suppose that $\langle U, y \rangle = U + Fy$. Then $\langle U, y \rangle$ is three dimensional and not simple (since U is two dimensional and abelian), and so solvable. Then, by using Corollary 2.4, we have that U contains a one-dimensional ideal K of $U + Fy$ such that $(U + Fy)/K$ is two-dimensional non-abelian, and $K = Z(\langle U, y \rangle)$.

Since U is maximal in $\langle U, x \rangle$ we have $\langle U, x \rangle \neq L$. Pick $y \in L \setminus \langle U, x \rangle$. Then $0 \neq Z(\langle U, x + y \rangle) \subseteq U$ by the above. Suppose $Z(\langle U, x \rangle) \neq Z(\langle U, y \rangle)$. Then $U = Z(\langle U, x \rangle) \oplus Z(\langle U, y \rangle)$. Let $0 \neq z \in Z(\langle U, x + y \rangle)$ and write $z = z_1 + z_2$ where $z_1 \in Z(\langle U, x \rangle)$, $z_2 \in Z(\langle U, y \rangle)$. Then $0 = [z, (x + y)] = [z_2, x] + [z_1, y]$, so $[z_2, x] = -[z_1, y]$. Now, if $z_1 = 0$, then $[z_2, x] = 0$, whence $z_2 \in Z(\langle U, x \rangle) \cap Z(\langle U, y \rangle)$, a contradiction. Similarly, if $z_2 = 0$, then $[z_1, y] = 0$, whence $z_1 \in Z(\langle U, x \rangle) \cap Z(\langle U, y \rangle)$, a contradiction again. Hence $z_1, z_2 \neq 0$. Since $z_1, z_2 \in U$ we deduce that $[z_1, y] = -[z_2, x] \in \langle U, x \rangle \cap \langle U, y \rangle = U$. Thus $y \in N_L(U) = U$, a contradiction. It follows that $Z(\langle U, x \rangle) = Z(\langle U, y \rangle)$ for all $y \in L$, whence $[L, Z(\langle U, x \rangle)] = 0$ and $Z(\langle U, x \rangle)$ is an ideal of L , contradicting the fact that U is core-free.

Next we establish analogues of two results of Varea from [15].

Theorem 4.2 *Let L be a Lie algebra over an algebraically closed field F of characteristic $p > 5$. If U is a sm subalgebra of L such that U/U_L is solvable and $\dim(U/U_L) > 1$, then U is modular in L , and hence L/U_L is isomorphic to $sl_2(F)$ or to a Zassenhaus algebra.*

Proof: Let L be a Lie algebra of minimal dimension having a sm subalgebra U which is not modular in L , and such that U/U_L is solvable and $\dim(U/U_L) > 1$. Then $U_L = 0$ and U is solvable. Since U is not a quasi-ideal there is an element $x \in L \setminus U$ such that $S = \langle U, x \rangle \neq U + Fx$. Let $K = U_S$. If $\dim(U/K) = 1$ then S/K is almost abelian, by Theorem 3.6, whence U is a quasi-ideal of S , a contradiction. It follows that $\dim(U/K) > 1$. If U/K is modular in S/K then $\dim(S/U) = 1$, by Theorem 2.4 of [15], a contradiction. The minimality of L then implies that $S = L$. This yields that U is modular in L , by Lemma 2.1. This contradiction establishes the result.

We say that the subalgebra U of L is *split* if $\text{ad}_L x$ is split for all $x \in U$; that is, if $\text{ad}_L x$ has a Jordan decomposition into semisimple and nilpotent parts for all $x \in U$.

Theorem 4.3 *Let L be a Lie algebra over a perfect field F of characteristic p different from 2. If U is a sm subalgebra of L which is split and which contains the normaliser of each of its non-zero subalgebras, then U is modular, and one of the following holds:*

- (i) L is almost abelian and $\dim(U) = 1$;
- (ii) $L \cong \text{sl}_2(F)$ and $\dim(U) = 2$;
- (iii) L is a Zassenhaus algebra and U is its unique subalgebra of codimension one in L .

Proof: Let L be a Lie algebra of minimal dimension having a sm subalgebra U which is split and which contains the normaliser of each of its non-zero subalgebras, but which is not modular in L . Since U is not a quasi-ideal there is an element $x \in L \setminus U$ such that $S = \langle U, x \rangle \neq U + Fx$. If $S \neq L$ then U is modular in S , by the minimality of L . It follows from Theorem 2.7 of [15] that U is a quasi-ideal of S , a contradiction. Hence $S = L$. Once again we see that U is modular in L , by Lemma 2.1. This contradiction establishes that U is modular in L . The result now follows from Theorem 2.7 of [15].

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